CS-150 Quantum Computer Science: Problem Set 1

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Spring, 2026

Guidelines: ***The deadline to return this problem set is 11.59pm on Friday February 27***. *Please show all work for full credit. You are welcome to collaborate with each other, but you must write your solutions independently. Use of AI is forbidden unless indicated otherwise. Submission of the problem set should be via Gradescope only. Best wishes!*

Problem 1 (Generalized Bloch Sphere) *In this problem, we generalize the Bloch sphere representation of single-qubit pure states to mixed states and multi-qubit systems.*

(a) ***Single-qubit density matrices.***

Show that any single-qubit density matrix can be written in the form

$$\rho = \frac{1}{2}\big(\mathbb{I} + \mathbf{v}\cdot\boldsymbol{\sigma}\big),$$

where $\boldsymbol{\sigma} = (X, Y, Z)$ *and* $\mathbf{v} = (v_x, v_y, v_z) \in \mathbb{R}^3$, *and*

$$\mathbf{v}\cdot\boldsymbol{\sigma} = v_x X + v_y Y + v_z Z.$$

Show that the coefficients are uniquely determined by

$$v_i = \mathrm{Tr}(\rho\sigma_i), \qquad i \in \{x, y, z\}.$$

(b) ***Purity.***

Show that

$$\mathrm{Tr}(\rho^2) = \frac{1}{2}\big(1 + \|\mathbf{v}\|^2\big).$$

Recall that $\mathrm{Tr}(\rho^2)$ *is called the* purity. *Conclude that*

$$\mathrm{Tr}(\rho^2) = 1 \quad \Longleftrightarrow \quad \|\mathbf{v}\| = 1,$$

and that the maximally mixed state corresponds to $\mathbf{v} = 0$.

(c) ***Geometric interpretation.***

Give a geometric interpretation of the set of single-qubit density matrices in terms of the vector $\mathbf{v}$. *Identify which points correspond to pure states and which correspond to mixed states.*

(d) ***Pauli basis for multi-qubit systems.***

Let $z = (\alpha, \beta) \in \mathbb{F}_2^{2n}$*, where* $\alpha, \beta \in \mathbb{F}_2^n$*. Define*

$$W(z) = i^{-\alpha\cdot\beta}\, X^\alpha Z^\beta,$$

where

$$X^\alpha = X_1^{\alpha_1} \otimes \cdots \otimes X_n^{\alpha_n}, \qquad Z^\beta = Z_1^{\beta_1} \otimes \cdots \otimes Z_n^{\beta_n},$$

and $\alpha \cdot \beta = \sum_{j=1}^n \alpha_j \beta_j \pmod 2$.

Show that the operators $\{W(z)\}_{z\in\mathbb{F}_2^{2n}}$ *form an orthonormal basis of the space of* $2^n \times 2^n$ *matrices with respect to the Hilbert–Schmidt inner product:*

$$\frac{1}{2^n}\operatorname{Tr}\big(W^\dagger(z_1)W(z_2)\big) = \delta_{z_1,z_2}.$$

(e) ***Generalized Bloch representation.***

Using the orthonormality above, show that any n-qubit density matrix can be written uniquely as

$$\rho = \frac{1}{2^n}\sum_{z\in\mathbb{F}_2^{2n}} r(z)W(z),$$

where

$$r(z) = \operatorname{Tr}(\rho W(z)) \in \mathbb{R}.$$

Show that

$$r(0) = 1$$

and prove that the purity satisfies

$$\operatorname{Tr}(\rho^2) = \frac{1}{2^n}\sum_{z\in\mathbb{F}_2^{2n}} r(z)^2.$$

Interpret this formula as stating that the purity is proportional to the squared Euclidean norm of the generalized Bloch vector $\{r(z)\}_{z\neq 0}$.

Problem 2 (Quantum Channels and Stinespring Dilation) *Let* $\Phi : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ *be a completely positive trace-preserving (CPTP) map. Recall that Stinespring's theorem states that there exists an auxiliary Hilbert space* $\mathcal{H}_E$ *and an isometry*

$$V : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$$

such that

$$\Phi(\rho) = \operatorname{Tr}_E\big(V\rho V^\dagger\big).$$

(a) ***From Stinespring to Kraus form.***

Let $\{|e_k\rangle\}$ *be an orthonormal basis of* $\mathcal{H}_E$*. Define operators* $K_k : \mathcal{H}_A \to \mathcal{H}_B$ *by*

$$K_k := \langle e_k|V.$$

Show that

$$\Phi(\rho) = \sum_k K_k \rho K_k^\dagger.$$

Show furthermore that trace preservation implies

$$\sum_k K_k^\dagger K_k = \mathbb{I}.$$

(b) ***From Kraus form to Stinespring.***

Conversely, suppose

$$\Phi(\rho) = \sum_{k=1}^{r} K_k \rho K_k^\dagger.$$

Construct explicitly an isometry

$$V : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathbb{C}^r$$

such that

$$\Phi(\rho) = \mathrm{Tr}_E\big(V \rho V^\dagger\big).$$

(c) ***Minimal dilation and Kraus rank.***

Show that the smallest possible dimension of the environment $\mathcal{H}_E$ equals the minimal number of Kraus operators needed to represent Φ.

(Hint: relate this to the rank of the Choi matrix.)

(d) ***Uniqueness of minimal dilation.***

Suppose

$$\Phi(\rho) = \mathrm{Tr}_E(V \rho V^\dagger) = \mathrm{Tr}_{E'}(V' \rho V'^\dagger)$$

are two minimal Stinespring dilations.

Show that there exists a unitary operator

$$U : \mathcal{H}_E \to \mathcal{H}_{E'}$$

such that

$$V' = (\mathbb{I} \otimes U) V.$$

(e) ***Unitary freedom in Kraus operators.***

Using part (d), prove that two Kraus representations

$$\{K_k\}_{k=1}^{r}, \qquad \{L_j\}_{j=1}^{r}$$

describe the same channel if and only if there exists a unitary matrix u_{jk} such that

$$L_j = \sum_k u_{jk} K_k.$$

(f) ***When is a channel unitary?***

Prove that Φ is a unitary channel (i.e., $\Phi(\rho) = U \rho U^\dagger$ for some unitary U) if and only if its minimal Kraus rank equals 1.

Problem 3 (Universality of 2 qubit gates) *Exercise 5.6 of Preskill* https://www.preskill.caltech.edu/ph219/chap5_15.pdf.

Problem 4 (Universality of the $\{H, T\}$ Gate Set) *In this problem, we prove that the gate set $\{H, T\}$ is* universal *for single-qubit quantum computation, i.e., the subgroup it generates is dense in* SU(2).

Any element of SU(2) *can be written as (a rotation)*

$$U = e^{-i\frac{\theta}{2}\hat{n}\cdot\boldsymbol{\sigma}},$$

where $\hat{n} = (n_x, n_y, n_z) \in \mathbb{R}^3$ satisfies $\|\hat{n}\| = 1$, and $\boldsymbol{\sigma} = (X, Y, Z)$. We denote by $R_X(\theta)$, $R_Y(\theta)$, and $R_Z(\theta)$ the rotations about the x-, y-, and z-axes, respectively.

(a) ***Generating rotations.***

Show that, up to global phase, T is the rotation $R_Z(\pi/4)$. Using the identity $HZH = X$, show that

$$HTH = R_X(\pi/4) \quad \textit{up to a global phase.}$$

Conclude that the gate set $\{H, T\}$ generates rotations by angle $\pi/4$ about two non-parallel axes.

(b) ***An irrational rotation.***

Consider the unitary

$$U = R_X(\pi/4) R_Z(\pi/4).$$

(i) *Compute* $\mathrm{Tr}(U)$.

(ii) *Using the identity*

$$\mathrm{Tr}\Big(e^{-i\frac{\theta}{2}\hat{n}\cdot\sigma}\Big) = 2\cos(\theta/2),$$

determine the rotation angle θ corresponding to U.

(iii) *Show that θ/π is irrational.*

Conclude that the subgroup generated by $\{H, T\}$ contains an element of infinite order.

Hint: *Express* $\cos(\theta/2)$ *explicitly and argue that it cannot equal* $\cos(\pi p/q)$ *for integers p, q.*

(c) ***Density in*** $\mathrm{SU}(2)$**.**

A structural theorem in Lie theory states that every closed *subgroup of* $\mathrm{SU}(2)$ *is either:*

- *finite,*
- *conjugate to a circle subgroup (rotations about a fixed axis), or*
- *all of* $\mathrm{SU}(2)$.

Let $G = \langle H, T \rangle$. Using parts (a) and (b), show that:

- *G is not finite,*
- *G is not contained in a circle subgroup.*

Conclude that the closure $\overline{G}$ equals $\mathrm{SU}(2)$. *Thus, $\{H, T\}$ is universal for single-qubit operations.*

Problem 5 (Quadratic Structure of Stabilizer States) *One way to characterize stabilizer states is as the family of quantum states that can be generated from the all-zero state $|0\rangle^{\otimes n}$ using Clifford operations. In this problem, we show that stabilizer states admit a* quadratic phase structure *in the computational basis.*

We will prove that any stabilizer state can be written in the form

$$|s\rangle = \frac{1}{\sqrt{|A|}} \sum_{x \in A} (-1)^{Q(x)} i^{\ell(x)} |x\rangle, \tag{1}$$

where:

- *$A \subseteq \mathbb{F}_2^n$ is an affine subspace, i.e.,*

  $$A = \{Ly + b : y \in \mathbb{F}_2^m\},$$

  for some matrix $L \in \mathbb{F}_2^{n\times m}$ of rank $m \leq n$ and $b \in \mathbb{F}_2^n$. In particular, $|A| = 2^m$.

- *$Q : \mathbb{F}_2^n \to \mathbb{F}_2$ is a quadratic function of the form*

  $$Q(x_1, \ldots, x_n) = \sum_{j<k} Q_{j,k} x_j x_k + \sum_j c_j x_j \pmod 2,$$

  where $Q_{j,k}, c_j \in \mathbb{F}_2$.

- $\ell : \mathbb{F}_2^n \to \mathbb{F}_2$ *is a linear function,*

$$\ell(x_1, \ldots, x_n) = \sum_j \ell_j x_j \pmod 2,$$

with $\ell_j \in \mathbb{F}_2$.

(a) ***Warm-up: the single-qubit case.***

Prove that every single-qubit stabilizer state can be written in the form (1).

Explicitly determine A*,* Q*, and* ℓ *for each of the six single-qubit stabilizer states*

$$|0\rangle\,,\ |1\rangle\,,\ |+\rangle\,,\ |-\rangle\,,\ |i+\rangle\,,\ |i-\rangle\,.$$

(b) ***Closure under Clifford generators.***

Suppose a state has the quadratic phase form (1)*. Show that after applying any of the following gates to the qubits, the state remains in the quadratic phase form over an affine subspace:*

$$\mathrm{CNOT}, \quad S, \quad H.$$

Using this closure property and induction on the number of Clifford gates, conclude that every stabilizer state has the form (1)*.*

Hint: *analyze the action of* CNOT*,* S *and* H *on the computational basis. Recall that the action of* CNOT *on the computational basis corresponds to a linear map. Recall* $S|x\rangle = i^x|x\rangle$*. Also, recall* H *corresponds to a Fourier transform over* $\mathbb{F}_2$*:*

$$H|x\rangle = \frac{1}{\sqrt{2}} \sum_{y \in \mathbb{F}_2} (-1)^{xy} |y\rangle\,.$$

(c) ***Normal form for Clifford operations.***

Show that for every Clifford operation C *(up to global phase), there exists another Clifford operation* C' *of the form*

$$C' = S'L'H',$$

where:

- H' *is a layer of Hadamard gates applied to a subset of qubits,*
- L' *is a layer consisting solely of* CNOT *and* X *gates,*
- S' *is a layer of* S *and* $C - Z$ *gates applied to a subset of qubits.*

Hint: *Using part (b)* $C|0^n\rangle$ *takes a normal form according to equation 1. Think about the Hadamard layer as creating a superposition over a linear subpspace,* CNOT *as creating a linear change of basis and* X *as adding a shift to the subpsace. The use* $C - Z$ *and* S *layer to instantiate* $(-1)^Q$ *and* i^ℓ*.*

Problem 6 (Extra credit) *Extend problem 4 and show that by including CNOT we can approximate arbitrary multi-qubit unitary operations to within arbitrary precision*